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Corporate Finance: Part I

Cost of Capital

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Contents

1. Introduction

This compendium provides a comprehensive overview of the most important topics covered in a corporate finance course at the Bachelor, Master or MBA level. The intension is to supplement renowned corporate finance textbooks such as Brealey, Myers and Allen's "Corporate Finance", Damodaran's "Corporate Finance - Theory and Practice", and Ross, Westerfield and Jordan's "Corporate Finance Fundamentals".

The compendium is designed such that it follows the structure of a typical corporate finance course. Throughout the compendium theory is supplemented with examples and illustrations.

2. The objective of the firm

Corporate Finance is about decisions made by corporations. Not all businesses are organized as corporations. Corporations have three distinct characteristics:

1. Corporations are legal entities, i.e. legally distinct from it owners and pay their own taxes
2. Corporations have limited liability, which means that shareholders can only loose their initial investment in case of bankruptcy
3. Corporations have separated ownership and control as owners are rarely managing the firm

The objective of the firm is to maximize shareholder value by increasing the value of the company's stock. Although other potential objectives (survive, maximize market share, maximize profits, etc.) exist these are consistent with maximizing shareholder value.

Most large corporations are characterized by separation of ownership and control. Separation of ownership and control occurs when shareholders not actively are involved in the management. The separation of ownership and control has the advantage that it allows share ownership to change without influencing with the day-to-day business. The disadvantage of separation of ownership and control is the agency problem, which incurs agency costs.

Agency costs are incurred when:

1. Managers do not maximize shareholder value
2. Shareholders monitor the management

In firms without separation of ownership and control (i.e. when shareholders are managers) no agency costs are incurred.

In a corporation the financial manager is responsible for two basic decisions:

1. The investment decision
2. The financing decision

The investment decision is what real assets to invest in, whereas the financing decision deals with how these investments should be financed. The job of the financial manager is therefore to decide on both such that shareholder value is maximized.

3. Present value and opportunity cost of capital

Present and future value calculations rely on the principle of time value of money.

> **Time value of money**
>
> One dollar today is worth more than one dollar tomorrow.

The intuition behind the time value of money principle is that one dollar today can start earning interest immediately and therefore will be worth more than one dollar tomorrow. Time value of money demonstrates that, all things being equal, it is better to have money now than later.

3.1 Compounded versus simple interest

When money is moved through time the concept of compounded interest is applied. Compounded interest occurs when interest paid on the investment during the first period is added to the principal. In the following period interest is paid on the new principal. This contrasts simple interest where the principal is constant throughout the investment period. To illustrate the difference between simple and compounded interest consider the return to a bank account with principal balance of €100 and an yearly interest rate of 5%. After 5 years the balance on the bank account would be:

- €125.0 with simple interest: €100 + 5 · 0.05 · €100 = €125.0
- €127.6 with compounded interest: €100 · 1.05^5 = €127.6

Thus, the difference between simple and compounded interest is the interest earned on interests. This difference is increasing over time, with the interest rate and in the number of sub-periods with interest payments.

3.2 Present value

Present value (PV) is the value today of a future cash flow. To find the present value of a future cash flow, C_t, the cash flow is multiplied by a discount factor:

(1) $$\text{PV} = \text{discount factor} \cdot C_t$$

The discount factor (DF) is the present value of €1 future payment and is determined by the rate of return on equivalent investment alternatives in the capital market.

(2) $$\text{DF} = \frac{1}{(1+r)^t}$$

Where r is the discount rate and t is the number of years. Inserting the discount factor into the present value formula yields:

(3) $$\text{PV} = \frac{C_t}{(1+\text{r})^{\text{t}}}$$

Example:

- What is the present value of receiving €250,000 two years from now if equivalent investments return 5%?

$$\text{PV} = \frac{C_t}{(1+\text{r})^{\text{t}}} = \frac{€250{,}000}{1.05^2} = €\ 226{,}757$$

- Thus, the present value of €250,000 received two years from now is €226,757 if the discount rate is 5 percent.

From time to time it is helpful to ask the inverse question: How much is €1 invested today worth in the future?. This question can be assessed with a future value calculation.

3.3 Future value

The future value (FV) is the amount to which an investment will grow after earning interest. The future value of a cash flow, C_0, is:

(4) $$FV = C_0 \cdot (1+r)^t$$

Example:

– What is the future value of €200,000 if interest is compounded annually at a rate of 5% for three years?

$$FV = €200{,}000 \cdot (1+.05)^3 = €231{,}525$$

- Thus, the future value in three years of €200,000 today is €231,525 if the discount rate is 5 percent.

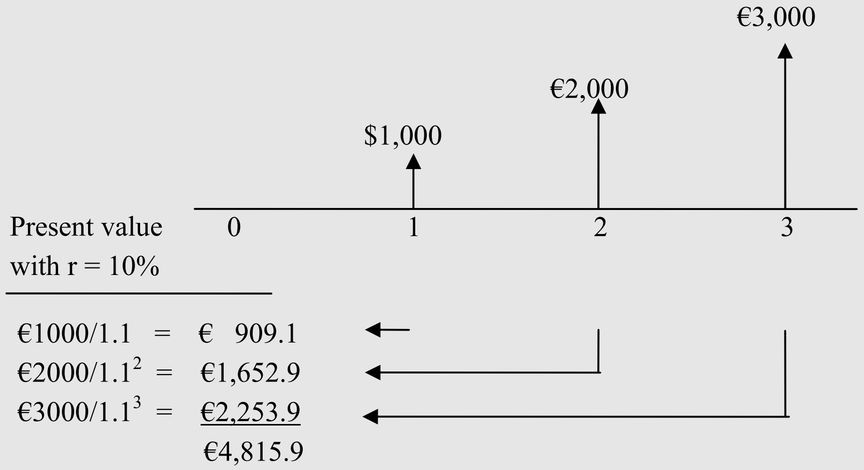
3.4 Principle of value additivity

The principle of value additivity states that present values (or future values) can be added together to evaluate multiple cash flows. Thus, the present value of a string of future cash flows can be calculated as the sum of the present value of each future cash flow:

(5) $$PV = \frac{C_1}{(1+r)^1} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + ... = \sum \frac{C_t}{(1+r)^t}$$

Example:

- The principle of value additivity can be applied to calculate the present value of the income stream of €1,000, €2000 and €3,000 in year 1, 2 and 3 from now, respectively.



- The present value of each future cash flow is calculated by discounting the cash flow with the 1, 2 and 3 year discount factor, respectively. Thus, the present value of €3,000 received in year 3 is equal to €3,000 / 1.1^3 = €2,253.9.
- Discounting the cash flows individually and adding them subsequently yields a present value of €4,815.9.

3.5 Net present value

Most projects require an initial investment. Net present value is the difference between the present value of future cash flows and the initial investment, C_0, required to undertake the project:

(6) $$\text{NPV} = \text{C}_0 + \sum_{i=1}^{n} \frac{C_i}{(1+r)^i}$$

Note that if C_0 is an initial investment, then $C_0 < 0$.

3.6 Perpetuities and annuities

Perpetuities and annuities are securities with special cash flow characteristics that allow for an easy calculation of the present value through the use of short-cut formulas.

Perpetuity

Security with a constant cash flow that is (theoretically) received forever. The present value of a perpetuity can be derived from the annual return, r, which equals the constant cash flow, C, divided by the present value (PV) of the perpetuity:

$$r = \frac{C}{PV}$$

Solving for PV yields:

(7) $$\text{PV of perpetuity} = \frac{C}{r}$$

Thus, the present value of a perpetuity is given by the constant cash flow, C, divided by the discount rate, r.

In case the cash flow of the perpetuity is growing at a constant rate rather than being constant, the present value formula is slightly changed. To understand how, consider the general present value formula:

$$PV = \frac{C_1}{(1+r)} + \frac{C_2}{(1+r)^2} + \frac{C_3}{(1+r)^3} + \cdots$$

Since the cash flow is growing at a constant rate g it implies that $C_2 = (1+g) \cdot C_1$, $C_3 = (1+g)^2 \cdot C_1$, etc. Substituting into the PV formula yields:

$$PV = \frac{C_1}{(1+r)} + \frac{(1+g)C_1}{(1+r)^2} + \frac{(1+g)^2 C_1}{(1+r)^3} + \cdots$$

Utilizing that the present value is a geometric series allows for the following simplification for the present value of growing perpetuity:

(8) $$\text{PV of growing perpetituity} = \frac{C_1}{r-g}$$

Annuity

An asset that pays a fixed sum each year for a specified number of years. The present value of an annuity can be derived by applying the principle of value additivity. By constructing two perpetuities, one with cash flows beginning in year 1 and one beginning in year t+1, the cash flow of the annuity beginning in year 1 and ending in year t is equal to the difference between the two perpetuities. By calculating the present value of the two perpetuities and applying the principle of value additivity, the present value of the annuity is the difference between the present values of the two perpetuities.

Asset	**Year of Payment** 0 1 2……….t t +1…………..	**Present Value**
Perpetuity 1 (first payment in year 1)		$\frac{C}{r}$
Perpetuity 2 (first payment in year t + 1)		$\left(\frac{C}{r}\right)\frac{1}{(1+r)^t}$
Annuity from (year 1 to year t)		$\left(\frac{C}{r}\right)-\left(\frac{C}{r}\right)\left(\frac{1}{(1+r)^t}\right)$

(9) $$\text{PV of annuity} = C\underbrace{\left[\frac{1}{r}-\frac{1}{r(1+r)^t}\right]}_{\text{Annuity factor}}$$

Note that the term in the square bracket is referred to as the annuity factor.

Example: Annuities in home mortgages

- When families finance their consumption the question often is to find a series of cash payments that provide a given value today, e.g. to finance the purchase of a new home. Suppose the house costs €300,000 and the initial payment is €50,000. With a 30-year loan and a monthly interest rate of 0.5 percent what is the appropriate monthly mortgage payment?

  The monthly mortgage payment can be found by considering the present value of the loan. The loan is an annuity where the mortgage payment is the constant cash flow over a 360 month period (30 years times 12 months = 360 payments):

  PV(loan) = mortgage payment · 360-monthly annuity factor

  Solving for the mortgage payment yields:

  $$\text{Mortgage payment} = \text{PV(Loan)/360-monthly annuity factor}$$
  $$= €250K / (1/0.005 – 1/(0.005 \cdot 1.005^{360})) = €1{,}498.87$$

  Thus, a monthly mortgage payment of €1,498.87 is required to finance the purchase of the house.

3.7 Nominal and real rates of interest

Cash flows can either be in current (nominal) or constant (real) dollars. If you deposit €100 in a bank account with an interest rate of 5 percent, the balance is €105 by the end of the year. Whether €105 can buy you more goods and services that €100 today depends on the rate of inflation over the year.

Inflation is the rate at which prices as a whole are increasing, whereas nominal interest rate is the rate at which money invested grows. The real interest rate is the rate at which the purchasing power of an investment increases.

The formula for converting nominal interest rate to a real interest rate is:

(10) $$1 + \text{real interest rate} = \frac{1+\text{nominal interest rate}}{1+\text{inflation rate}}$$

For small inflation and interest rates the real interest rate is approximately equal to the nominal interest rate minus the inflation rate.

Investment analysis can be done in terms of real or nominal cash flows, but discount rates have to be defined consistently

– Real discount rate for real cash flows
– Nominal discount rate for nominal cash flows

3.8 Valuing bonds using present value formulas

A bond is a debt contract that specifies a fixed set of cash flows which the issuer has to pay to the bondholder. The cash flows consist of a coupon (interest) payment until maturity as well as repayment of the par value of the bond at maturity.

The value of a bond is equal to the present value of the future cash flows:

(11) Value of bond = PV(cash flows) = PV(coupons) + PV(par value)

Since the coupons are constant over time and received for a fixed time period the present value can be found by applying the annuity formula:

(12) PV(coupons) = coupon · annuity factor

Example:

- Consider a 10-year US government bond with a par value of $1,000 and a coupon payment of $50. What is the value of the bond if other medium-term US bonds offered a 4% return to investors?

  Value of bond = PV(Coupon) + PV(Par value)
  = $\$50 \cdot [1/0.04 - 1/(0.04 \cdot 1.04^{10})] + \$1{,}000 \cdot 1/1.04^{10}$
  = $\$50 \cdot 8.1109 + \$675.56 = \$1{,}081.1$

  Thus, if other medium-term US bonds offer a 4% return to investors the price of the 10-year government bond with a coupon interest rate of 5% is $1,081.1.

The rate of return on a bond is a mix of the coupon payments and capital gains or losses as the price of the bond changes:

(13) $$\text{Rate of return on bond} = \frac{\text{coupon income} + \text{price change}}{\text{investment}}$$

Because bond prices change when the interest rate changes, the rate of return earned on the bond will fluctuate with the interest rate. Thus, the bond is subject to interest rate risk. All bonds are not equally affected by interest rate risk, since it depends on the sensitivity to interest rate fluctuations.

The interest rate required by the market on a bond is called the bond's yield to maturity. Yield to maturity is defined as the discount rate that makes the present value of the bond equal to its price. Moreover, yield to maturity is the return you will receive if you hold the bond until maturity. Note that the yield to maturity is different from the rate of return, which measures the return for holding a bond for a specific time period.

To find the yield to maturity (rate of return) we therefore need to solve for r in the price equation.

Example:

- What is the yield to maturity of a 3-year bond with a coupon interest rate of 10% if the current price of the bond is 113.6?

  Since yield to maturity is the discount rate that makes the present value of the future cash flows equal to the current price, we need to solve for r in the equation where price equals the present value of cash flows:

$$\text{PV(Cash flows)} = \text{Price on bond}$$

$$\frac{10}{(1+r)} + \frac{10}{(1+r)^2} + \frac{110}{(1+r)^3} = 113.6$$

  The yield to maturity is the found by solving for r by making use of a spreadsheet, a financial calculator or by hand using a trail and error approach.

$$\frac{10}{1.05} + \frac{10}{1.05^2} + \frac{110}{1.05^3} = 113.6$$

  Thus, if the current price is equal to 113.6 the bond offers a return of 5 percent if held to maturity.

The yield curve is a plot of the relationship between yield to maturity and the maturity of bonds.

Figure 1: Yield curve

As illustrated in Figure 1 the yield curve is (usually) upward sloping, which means that long-term bonds have higher yields. This happens because long-term bonds are subject to higher interest rate risk, since long-term bond prices are more sensitive to changes to the interest rate.

The yield to maturity required by investors is determined by

1. Interest rate risk
2. Time to maturity
3. Default risk

The default risk (or credit risk) is the risk that the bond issuer may default on its obligations. The default risk can be judged from credit ratings provided by special agencies such as Moody's and Standard and Poor's. Bonds with high credit ratings, reflecting a strong ability to repay, are referred to as investment grade, whereas bonds with a low credit rating are called speculative grade (or junk bonds).

In summary, there exist five important relationships related to a bond's value:

1. The value of a bond is **reversely** related to changes in the interest rate
2. Market value of a bond will be **less** than par value if investor’s required rate is **above** the coupon interest rate

3. As maturity approaches the market value of a bond approaches par value
4. Long-term bonds have **greater** interest rate risk than do short-term bonds
5. Sensitivity of a bond's value to changing interest rates depends **not only** on the length of time to maturity, but also on the patterns of cash flows provided by the bond

3.9 Valuing stocks using present value formulas

The price of a stock is equal to the present value of all future dividends. The intuition behind this insight is that the cash payoff to owners of the stock is equal to cash dividends plus capital gains or losses. Thus, the expected return that an investor expects from a investing in a stock over a set period of time is equal to:

$$(14) \qquad \text{Expected return on stock} = \text{r} = \frac{\text{dividend} + \text{capital gain}}{\text{investment}} = \frac{Div_1 + P_1 - P_0}{P_0}$$

Where Div_t and P_t denote the dividend and stock price in year t, respectively. Isolating the current stock price P_0 in the expected return formula yields:

$$(15) \qquad P_0 = \frac{Div_1 + P_1}{1+r}$$

The question then becomes "What determines next years stock price P_1?". By changing the subscripts next year's price is equal to the discounted value of the sum of dividends and expected price in year 2:

$$P_1 = \frac{Div_2 + P_2}{1+r}$$

Inserting this into the formula for the current stock price P_0 yields:

$$P_0 = \frac{Div_1 + P_1}{1+r} = \frac{1}{1+r}\left(Div_1 + P_1\right) = \frac{1}{1+r}\left(Div_1 + \frac{Div_2 + P_2}{1+r}\right) = \frac{Div_1}{1+r} + \frac{Div_2 + P_2}{(1+r)^2}$$

By recursive substitution the current stock price is equal to the sum of the present value of all future dividends plus the present value of the horizon stock price, P_H.

$$
\begin{aligned}
P_0 &= \frac{Div_1}{1+r} + \frac{Div_2}{(1+r)^2} + \frac{Div_3 + P_3}{(1+r)^3} \\
&\vdots \\
P_0 &= \frac{Div_1}{1+r} + \frac{Div_2}{(1+r)^2} + \cdots + \frac{Div_H + P_H}{(1+r)^H} \\
&= \sum_{t=1}^{H} \frac{Div_t}{(1+r)^t} + \frac{P_H}{(1+r)^H}
\end{aligned}
$$

The final insight is that as H approaches zero, $[P_H / (1+r)^H]$ approaches zero. Thus, in the limit the current stock price, P_0, can be expressed as the sum of the present value of all future dividends.

Discounted dividend model

(16) $$P_0 = \sum_{t=1}^{\infty} \frac{Div_t}{(1+r)^t}$$

In cases where firms have constant growth in the dividend a special version of the discounted dividend model can be applied. If the dividend grows at a constant rate, g, the present value of the stock can be found by applying the present value formula for perpetuities with constant growth.

Discounted dividend growth model

(17) $$P_0 = \frac{Div_1}{r-g}$$

The discounted dividend growth model is often referred to as the Gordon growth model.

Some firms have both common and preferred shares. Common stockholders are residual claimants on corporate income and assets, whereas preferred shareholders are entitled only to a fixed dividend (with priority over common stockholders). In this case the preferred stocks can be valued as a perpetuity paying a constant dividend forever.

(18) $$P_0 = \frac{Div}{r}$$

The perpetuity formula can also be applied to value firms in general if we assume no growth and that all earnings are paid out to shareholders.

(19) $$P_0 = \frac{Div_1}{r} = \frac{EPS_1}{r}$$

If a firm elects to pay a lower dividend, and reinvest the funds, the share price may increase because future dividends may be higher.

Growth can be derived from applying the return on equity to the percentage of earnings ploughed back into operations:

(20) g = return on equity · plough back ratio

Where the plough back ratio is the fraction of earnings retained by the firm. Note that the plough back ratio equals (1 - payout ratio), where the payout ratio is the fraction of earnings paid out as dividends.

The value of growth can be illustrated by dividing the current stock price into a non-growth part and a part related to growth.

(21) $$P_{\text{With growth}} = P_{\text{No growth}} + PVGO$$

Where the growth part is referred to as the present value of growth opportunities (PVGO). Inserting the value of the no growth stock from (22) yields:

(22) $$P_0 = \frac{EPS_1}{r} + PVGO$$

Firms in which PVGO is a substantial fraction of the current stock price are referred to as growth stocks, whereas firms in which PVGO is an insignificant fraction of the current stock prices are called income stocks.

4. The net present value investment rule

Net present value is the difference between a project's value and its costs. The net present value investment rule states that firms should only invest in projects with positive net present value.

When calculating the net present value of a project the appropriate discount rate is the opportunity cost of capital, which is the rate of return demanded by investors for an equally risky project. Thus, the net present value rule recognizes the time value of money principle.

To find the net present value of a project involves several steps:

How to find the net present value of a project

1. Forecast cash flows
2. Determinate the appropriate opportunity cost of capital, which takes into account the principle of time value of money and the risk-return trade-off
3. Use the discounted cash flow formula and the opportunity cost of capital to calculate the present value of the future cash flows
4. Find the net present value by taking the difference between the present value of future cash flows and the project's costs

There exist several other investment rules:

- Book rate of return
- Payback rule
- Internal rate of return

To understand why the net present value rule leads to better investment decisions than the alternatives it is worth considering the desirable attributes for investment decision rules. The goal of the corporation is to maximize firm value. A shareholder value maximizing investment rule is:

- Based on cash flows
- Taking into account time value of money
- Taking into account differences in risk

The net present value rule meets all these requirements and directly measures the value for shareholders created by a project. This is fare from the case for several of the alternative rules.

The book rate of return is based on accounting returns rather than cash flows:

> **Book rate of return**
>
> Average income divided by average book value over project life
>
> (23) $$\text{Book rate of return} = \frac{\text{book income}}{\text{book value of assets}}$$

The main problem with the book rate of return is that it only includes the annual depreciation charge and not the full investment. Due to time value of money this provides a negative bias to the cost of the investment and, hence, makes the return appear higher. In addition no account is taken for risk. Due to the risk return trade-off we might accept poor high risk projects and reject good low risk projects.

> **Payback rule**
>
> The payback period of a project is the number of years it takes before the cumulative forecasted cash flow equals the initial outlay.

The payback rule only accepts projects that "payback" in the desired time frame.

This method is flawed, primarily because it ignores later year cash flows and the present value of future cash flows. The latter problem can be solved by using a payback rule based on discounted cash flows.

Internal rate of return (IRR)

Defined as the rate of return which makes NPV=0. We find IRR for an investment project lasting T years by solving:

(24) $$NPV = C_o + \frac{C_1}{1+IRR} + \frac{C_2}{(1+IRR)^2} + \cdots + \frac{C_T}{(1+IRR)^T} = 0$$

The IRR investment rule accepts projects if the project's IRR exceeds the opportunity cost of capital, i.e. when IRR > r.

Finding a project's IRR by solving for NPV equal to zero can be done using a financial calculator, spreadsheet or trial and error calculation by hand.

Mathematically, the IRR investment rule is equivalent to the NPV investment rule. Despite this the IRR investment rule faces a number of pitfalls when applied to projects with special cash flow characteristics.

1. Lending or borrowing?
   - With certain cash flows the NPV of the project ***increases*** if the discount rate ***increases***. This is contrary to the normal relationship between NPV and discount rates
2. Multiple rates of return
   - Certain cash flows can generate NPV=0 at multiple discount rates. This will happen when the cash flow stream changes sign. Example: Maintenance costs. In addition, it is possible to have projects with ***no*** IRR and a ***positive*** NPV
3. Mutually exclusive projects
   - Firms often have to choose between mutually exclusive projects. IRR sometimes ignores the magnitude of the project. Large projects with a lower IRR might be preferred to small projects with larger IRR.
4. Term structure assumption
   - We assume that discount rates are constant for the term of the project. What do we compare the IRR with, if we have different rates for each period, *r1, r2, r3*, …? It is not easy to find a traded security with equivalent risk and the same time pattern of cash flows

Finally, note that both the IRR and the NPV investment rule are discounted cash flow methods. Thus, both methods possess the desirable attributes for an investment rule, since they are based on cash flows and allows for risk and time value of money. Under careful use both methods give the same investment decisions (whether to accept or reject a project). However, they may not give the same ranking of projects, which is a problem in case of mutually exclusive projects.

5. Risk, return and opportunity cost of capital

Opportunity cost of capital depends on the risk of the project. Thus, to be able to determine the opportunity cost of capital one must understand how to measure risk and how investors are compensated for taking risk.

5.1 Risk and risk premia

The risk premium on financial assets compensates the investor for taking risk. The risk premium is the difference between the return on the security and the risk free rate.

To measure the average rate of return and risk premium on securities one has to look at very long time periods to eliminate the potential bias from fluctuations over short intervals.

Over the last 100 years U.S. common stocks have returned an average annual nominal compounded rate of return of 10.1% compared to 4.1% for U.S. Treasury bills. As U.S. Treasury bill has short maturity and there is no risk of default, short-term government debt can be considered risk-free. Investors in common stocks have earned a risk premium of 7.0 percent (10.1 - 4.1 percent.). Thus, on average investors in common stocks have historically been compensated with a 7.0 percent higher return per year for taking on the risk of common stocks.

Table 1: Average nominal compounded returns, standard deviation and risk premium on U.S. securities, 1900-2000.

	Annual return	Std. variation	Risk premium
U.S. Treasury Bills	4.1%	4.7%	0.0%
U.S. Government Bonds	4.8%	10.0%	0.7%
U.S. Common Stocks	10.1%	20.2%	7.0%

Source: E. Dimson, P.R. Mash, and M Stauton, Triumph of the Optimists: 101 Years of Investment returns, Princeton University Press, 2002.

Across countries the historical risk premium varies significantly. In Denmark the average risk premium was only 4.3 percent compared to 10.7 percent in Italy. Some of these differences across countries may reflect differences in business risk, while others reflect the underlying economic stability over the last century.

The historic risk premium may overstate the risk premium demanded by investors for several reasons. First, the risk premium may reflect the possibility that the economic development could have turned out to be less fortunate. Second, stock returns have for several periods outpaced the underlying growth in earnings and dividends, something which cannot be expected to be sustained.

The risk of financial assets can be measured by the spread in potential outcomes. The variance and standard deviation on the return are standard statistical measures of this spread.

Variance

Expected (average) value of squared deviations from mean. The variance measures the return volatility and the units are percentage squared.

(25) $$\text{Variance}(r) = \sigma^2 = \frac{1}{N-1}\sum_{t=1}^{N}(r_t - \bar{r})^2$$

Where $\bar{r}$ denotes the average return and N is the total number of observations.

Standard deviation

Square root of variance. The standard deviation measures the return volatility and units are in percentage.

(26) $$\text{Std.dev.}(r) = \sqrt{\text{variance}(r)} = \sigma$$

Using the standard deviation on the yearly returns as measure of risk it becomes clear that U.S. Treasury bills were the least variable security, whereas common stock were the most variable. This insight highlights the risk-return tradeoff, which is key to the understanding of how financial assets are priced.

Risk-return tradeoff
Investors will not take on additional risk unless they expect to be compensated with additional return

The risk-return tradeoff relates the expected return of an investment to its risk. Low levels of uncertainty (low risk) are associated with low expected returns, whereas high levels of uncertainty (high risk) are associated with high expected returns.

It follows from the risk-return tradeoff that rational investors will when choosing between two assets that offer the same expected return prefer the less risky one. Thus, an investor will take on increased risk only if compensated by higher expected returns. Conversely, an investor who wants higher returns must accept more risk. The exact trade-off will differ by investor based on individual risk aversion characteristics (i.e. the individual preference for risk taking).

5.2 The effect of diversification on risk

The risk of an individual asset can be measured by the variance on the returns. The risk of individual assets can be reduced through diversification. Diversification reduces the variability when the prices of individual assets are not perfectly correlated. In other words, investors can reduce their exposure to individual assets by holding a diversified portfolio of assets. As a result, diversification will allow for the same portfolio return with reduced risk.

Example:

- A classical example of the benefit of diversification is to consider the effect of combining the investment in an ice-cream producer with the investment in a manufacturer of umbrellas. For simplicity, assume that the return to the ice-cream producer is +15% if the weather is sunny and -10% if it rains. Similarly the manufacturer of umbrellas benefits when it rains (+15%) and looses when the sun shines (-10%). Further, assume that each of the two weather states occur with probability 50%.

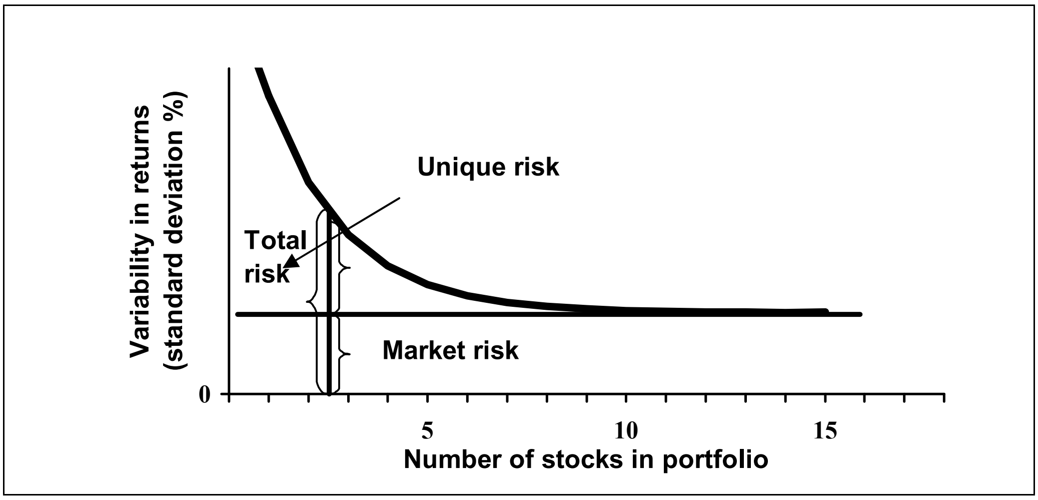
	Expected return	**Variance**
Ice-cream producer	0.5·15% + 0.5·-10% = 2.5%	$0.5\cdot[15-2.5]^2 + 0.5\cdot[-10-2.5]^2 = 12.5^2\%$
Umbrella manufacturer	0.5·-10% + 0.5·15% = 2.5%	$0.5\cdot[-10-2.5]^2 + 0.5\cdot[15-2.5]^2 = 12.5^2\%$

- Both investments offer an expected return of +2.5% with a standard deviation of 12.5 percent

- Compare this to the portfolio that invests 50% in each of the two stocks. In this case, the expected return is +2.5% both when the weather is sunny and rainy (0.5*15% + 0.5*-10% = 2.5%). However, the standard deviation drops to 0% as there is no variation in the return across the two states. Thus, by diversifying the risk related to the weather could be hedged. This happens because the returns to the ice-cream producer and umbrella manufacturer are perfectly negatively correlated.

Obviously the prior example is extreme as in the real world it is difficult to find investments that are perfectly negatively correlated and thereby diversify away all risk. More generally the standard deviation of a portfolio is reduced as the number of securities in the portfolio is increased. The reduction in risk will occur if the stock returns within our portfolio are not perfectly positively correlated. The benefit of diversification can be illustrated graphically:

Figure 2: How portfolio diversification reduces risk



As the number of stocks in the portfolio increases the exposure to risk decreases. However, portfolio diversification cannot eliminate all risk from the portfolio. Thus, total risk can be divided into two types of risk: (1) Unique risk and (2) Market risk. It follows from the graphically illustration that unique risk can be diversified way, whereas market risk is non-diversifiable. Total risk declines until the portfolio consists of around 15-20 securities, then for each additional security in the portfolio the decline becomes very slight.

Portfolio risk

Total risk = Unique risk + Market risk

Unique risk

- Risk factors affecting only a single assets or a small group of assets
- Also called
    - Idiosyncratic risk
    - Unsystematic risk
    - Company-unique risk
    - Diversifiable risk
    - Firm specific risk
- Examples:
    - A strike among the workers of a company, an increase in the interest rate a company pays on its short-term debt by its bank, a product liability suit.

Market risk

- Economy-wide sources of risk that affects the overall stock market. Thus, market risk influences a large number of assets, each to a greater or lesser extent.
- Also called
    - Systematic risk
    - Non-diversifiable risk
- Examples:
    - Changes in the general economy or major political events such as changes in general interest rates, changes in corporate taxation, etc.

As diversification allows investors to essentially eliminate the unique risk, a well-diversified investor will only require compensation for bearing the market risk of the individual security. Thus, the expected return on an asset depends only on the market risk.

5.3 Measuring market risk

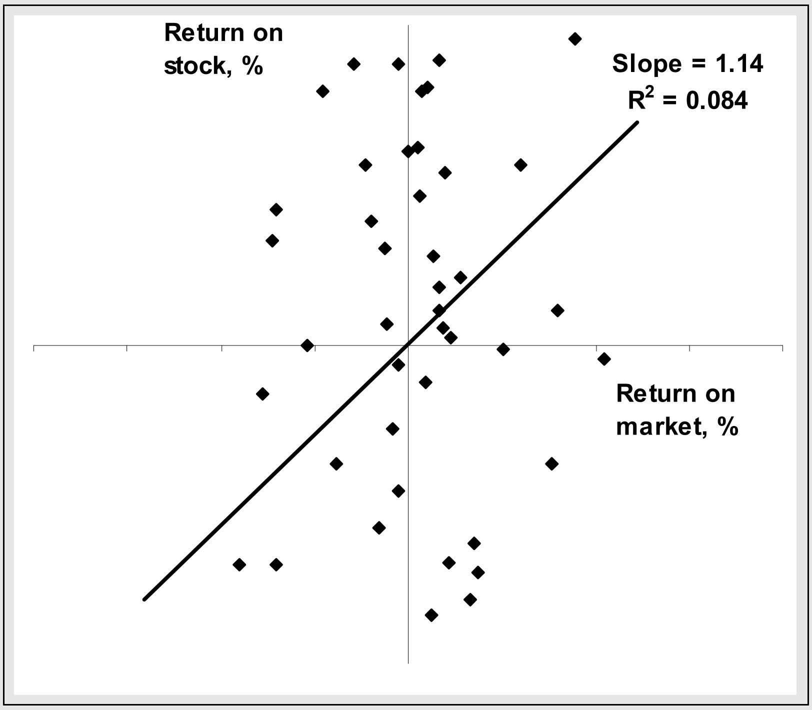
Market risk can be measured by beta, which measures how sensitive the return is to market movements. Thus, beta measures the risk of an asset relative to the average asset. By definition the average asset has a beta of one relative to itself. Thus, stocks with betas below 1 have lower than average market risk; whereas a beta above 1 means higher market risk than the average asset.

Estimating beta

Beta is measuring the individual asset's exposure to market risk. Technically the beta on a stock is defined as the covariance with the market portfolio divided by the variance of the market:

(27) $$\beta_i = \frac{\text{covariance with market}}{\text{variance of market}} = \frac{\sigma_{im}}{\sigma_m^2}$$

In practise the beta on a stock can be estimated by fitting a line to a plot of the return to the stock against the market return. The standard approach is to plot monthly returns for the stock against the market over a 60-month period.



Intuitively, beta measures the average change to the stock price when the market rises with an extra percent. Thus, beta is the slope on the fitted line, which takes the value 1.14 in the example above. A beta of 1.14 means that the stock amplifies the movements in the stock market, since the stock price will increase with 1.14% when the market rise an extra 1%. In addition it is worth noticing that r-square is equal to 8.4%, which means that only 8.4% of the variation in the stock price is related to market risk.

5.4 Portfolio risk and return

The expected return on a portfolio of stocks is a weighted average of the expected returns on the individual stocks. Thus, the expected return on a portfolio consisting of n stocks is:

(28) $$\text{Portfolio return} = \sum_{i=1}^{n} w_i r_i$$

Where w_i denotes the fraction of the portfolio invested in stock *i* and r_i is the expected return on stock *i*.

Example:

- Suppose you invest 50% of your portfolio in Nokia and 50% in Nestlé. The expected return on your Nokia stock is 15% while Nestlé offers 10%. What is the expected return on your portfolio?
- $\text{Portfolio return} = \sum_{i=1}^{n} w_i r_i = 0.5 \cdot 15\% + 0.5 \cdot 10\% = 12.5\%$
- A portfolio with 50% invested in Nokia and 50% in Nestlé has an expected return of 12.5%.

5.4.1 Portfolio variance

Calculating the variance on a portfolio is more involved. To understand how the portfolio variance is calculated consider the simple case where the portfolio only consists of two stocks, stock 1 and 2. In this case the calculation of variance can be illustrated by filling out four boxes in the table below.

Table 2: Calculation of portfolio variance

	Stock 1	Stock 2
Stock 1	$w_1^2\sigma_1^2$	$w_1 w_2 \sigma_{12} = w_1 w_2 \rho_{12}\sigma_1\sigma_2$
Stock 2	$w_1 w_2 \sigma_{12} = w_1 w_2 \rho_{12}\sigma_1\sigma_2$	$w_2^2\sigma_2^2$

In the top left corner of Table 2, you weight the variance on stock 1 by the square of the fraction of the portfolio invested in stock 1. Similarly, the bottom left corner is the variance of stock 2 times the square of the fraction of the portfolio invested in stock 2. The two entries in the diagonal boxes depend on the covariance between stock 1 and 2. The covariance is equal to the correlation coefficient times the product of the two standard deviations on stock 1 and 2. The portfolio variance is obtained by adding the content of the four boxes together:

$$\text{Portfolio variance} = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \rho_{12}\sigma_1\sigma_2$$

The benefit of diversification follows directly from the formula of the portfolio variance, since the portfolio variance is increasing in the covariance between stock 1 and 2. Combining stocks with a low correlation coefficient will therefore reduce the variance on the portfolio.

Example:

- Suppose you invest 50% of your portfolio in Nokia and 50% in Nestlé. The standard deviation on Nokia's and Nestlé's return is 30% and 20%, respectively. The correlation coefficient between the two stocks is 0.4. What is the portfolio variance?

$$\begin{aligned}\text{Portfolio variance} &= w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1w_2\rho_{12}\sigma_1\sigma_2 \\ &= 0.5^2 \cdot 30^2 + 0.5^2 20^2 + 2 \cdot 0.5 \cdot 0.5 \cdot 0.4 \cdot 30 \cdot 20 \\ &= 445 = 21.1^2\end{aligned}$$

- A portfolio with 50% invested in Nokia and 50% in Nestlé has a variance of 445, which is equivalent to a standard deviation of 21.1%.

For a portfolio of n stocks the portfolio variance is equal to:

(29) $$\text{Portfolio variance} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}$$

Note that when i=j, σ_{ij} is the variance of stock *i*, σ_i^2. Similarly, when i≠j, σ_{ij} is the covariance between stock i and j as $\sigma_{ij} = \rho_{ij}\sigma_i\sigma_j$.

5.4.2 Portfolio's market risk

The market risk of a portfolio of assets is a simple weighted average of the betas on the individual assets.

(30) $$\text{Portfolio beta} = \sum_{i=1}^{n} \mathrm{w}_i \beta_i$$

Where w_i denotes the fraction of the portfolio invested in stock *i* and β_i is market risk of stock *i*.

Example:

- Consider the portfolio consisting of three stocks A, B and C.

	Amount invested	Expected return	Beta
Stock A	1000	10%	0.8
Stock B	1500	12%	1.0
Stock C	2500	14%	1.2

- What is the beta on this portfolio?
- As the portfolio beta is a weighted average of the betas on each stock, the portfolio weight on each stock should be calculated. The investment in stock A is \$1000 out of the total investment of \$5000, thus the portfolio weight on stock A is 20%, whereas 30% and 50% are invested in stock B and C, respectively.
- The expected return on the portfolio is:

$$r_P = \sum_{i=1}^{n} w_i r_i = 0.2 \cdot 10\% + 0.3 \cdot 12\% + 0.5 \cdot 14\% = 12.6\%$$

- Similarly, the portfolio beta is:

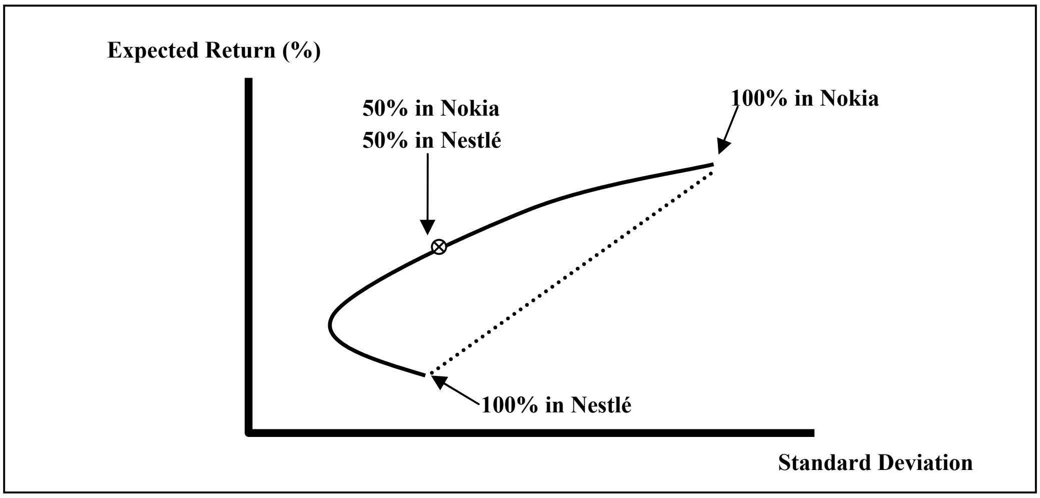
$$\beta_P = \sum_{i=1}^{n} w_i \beta_i = 0.2 \cdot 0.8 + 0.3 \cdot 1 + 0.5 \cdot 1.2 = 1.06$$

- The portfolio investing 20% in stock A, 30% in stock B, and 50% in stock C has an expected return of 12.6% and a beta of 1.06. Note that a beta above 1 implies that the portfolio has greater market risk than the average asset.

5.5 Portfolio theory

Portfolio theory provides the foundation for estimating the return required by investors for different assets. Through diversification the exposure to risk could be minimized, which implies that portfolio risk is less than the average of the risk of the individual stocks. To illustrate this consider Figure 3, which shows how the expected return and standard deviation change as the portfolio is comprised by different combinations of the Nokia and Nestlé stock.

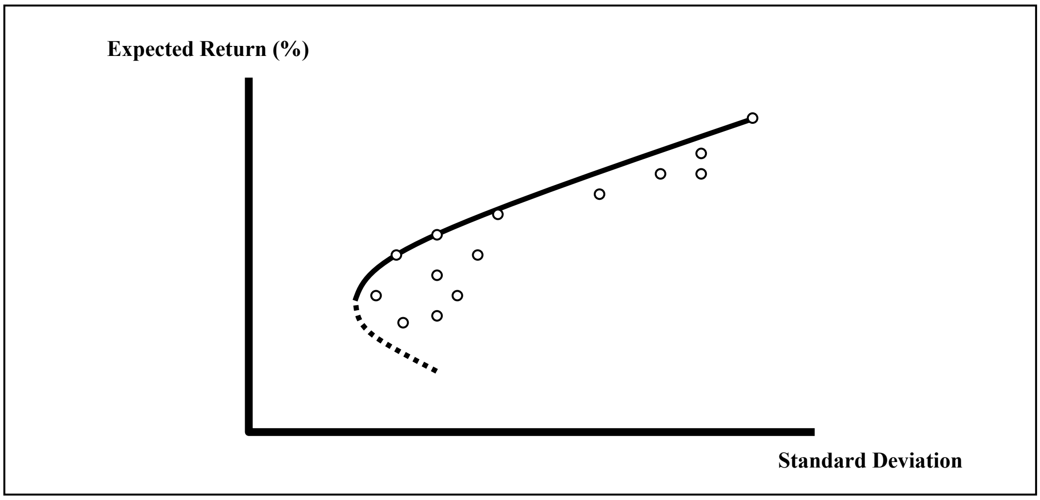
Figure 3: Portfolio diversification

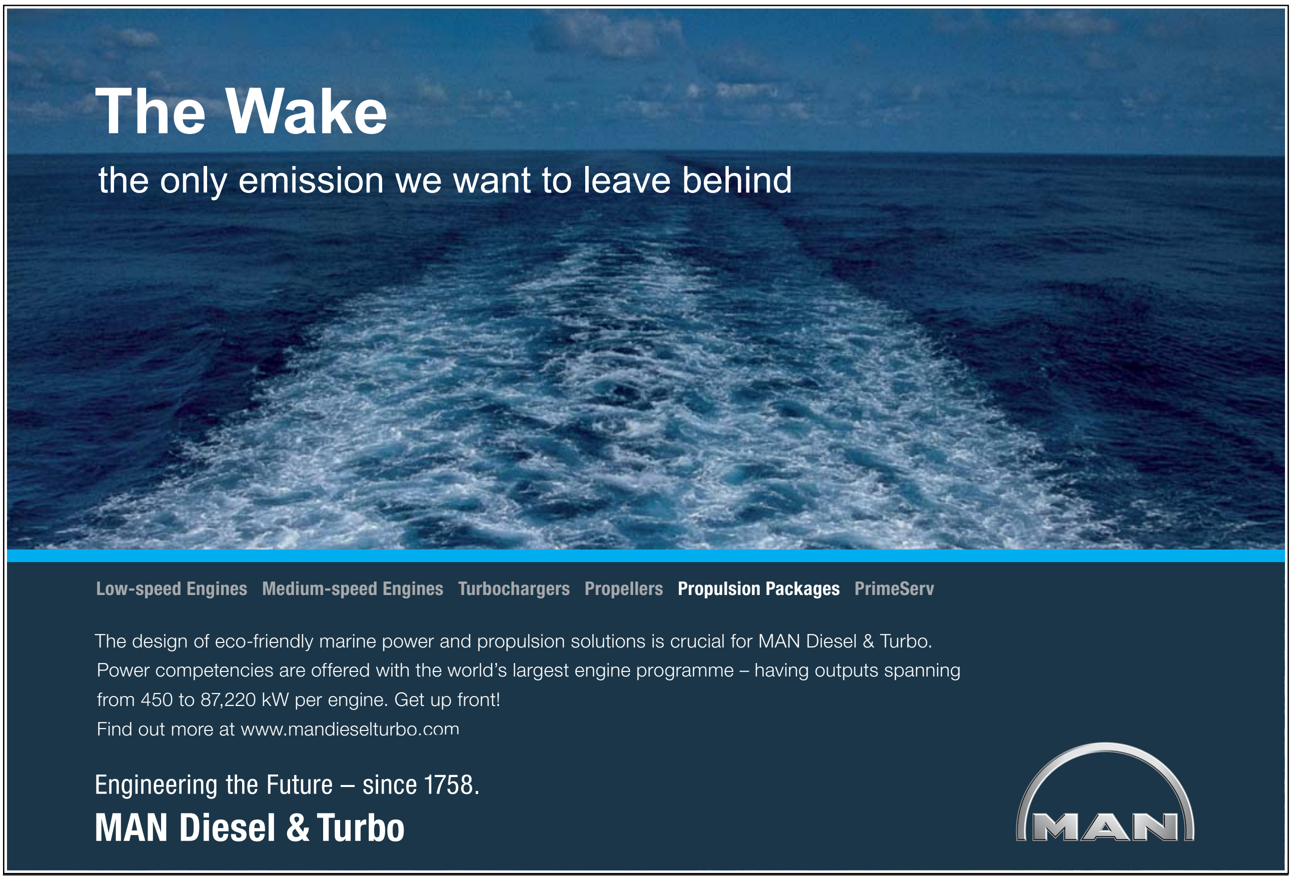


If the portfolio invested 100% in Nestlé the expected return would be 10% with a standard deviation of 20%. Similarly, if the portfolio invested 100% in Nokia the expected return would be 15% with a standard deviation of 30%. However, a portfolio investing 50% in Nokia and 50% in Nestlé would have an expected return of 12.5% with a standard deviation of 21.1%. Note that the standard deviation of 21.1% is less than the average of the standard deviation of the two stocks ($0.5 \cdot 20\% + 0.5 \cdot 30\% = 25\%$). This is due to the benefit of diversification.

In similar vein, every possible asset combination can be plotted in risk-return space. The outcome of this plot is the collection of all such possible portfolios, which defines a region in the risk-return space. As the objective is to minimize the risk for a given expected return and maximize the expected return for a given risk, it is preferred to move up and to the left in Figure 4.

Figure 4: Portfolio theory and the efficient frontier





The solid line along the upper edge of this region is known as the *efficient frontier.* Combinations along this line represent portfolios for which there is lowest risk for a given level of return. Conversely, for a given amount of risk, the portfolio lying on the efficient frontier represents the combination offering the best possible return. Thus, the efficient frontier is a collection of portfolios, each one optimal for a given amount of risk.

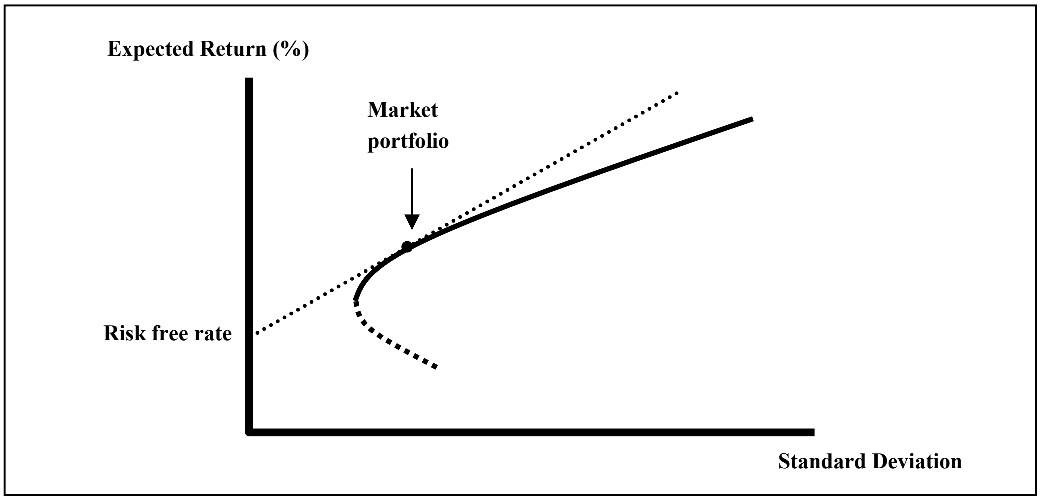
The Sharpe-ratio measures the amount of return above the risk-free rate a portfolio provides compared to the risk it carries.

(31) $$\text{Sharpe ratio on portfolio i} = \frac{r_i - r_f}{\sigma_i}$$

Where r_i is the return on portfolio *i*, r_f is the risk free rate and σ_i is the standard deviation on portfolio *i*'s return. Thus, the Sharpe-ratio measures the risk premium on the portfolio per unit of risk.

In a well-functioning capital market investors can borrow and lend at the same rate. Consider an investor who borrows and invests fraction of the funds in a portfolio of stocks and the rest in short-term government bonds. In this case the investor can obtain an expected return from such an allocation along the line from the risk free rate r_f through the tangent portfolio in Figure 5. As lending is the opposite of borrowing the line continues to the right of the tangent portfolio, where the investor is borrowing additional funds to invest in the tangent portfolio. This line is known as the capital allocation line and plots the expected return against risk (standard deviation).

Figure 5: Portfolio theory



The tangent portfolio is called the market portfolio. The market portfolio is the portfolio on the efficient frontier with the highest Sharpe-ratio. Investors can therefore obtain the best possible risk return trade-off by holding a mixture of the market portfolio and borrowing or lending. Thus, by combining a risk-free asset with risky assets, it is possible to construct portfolios whose risk-return profiles are superior to those on the efficient frontier.

5.6 Capital assets pricing model (CAPM)

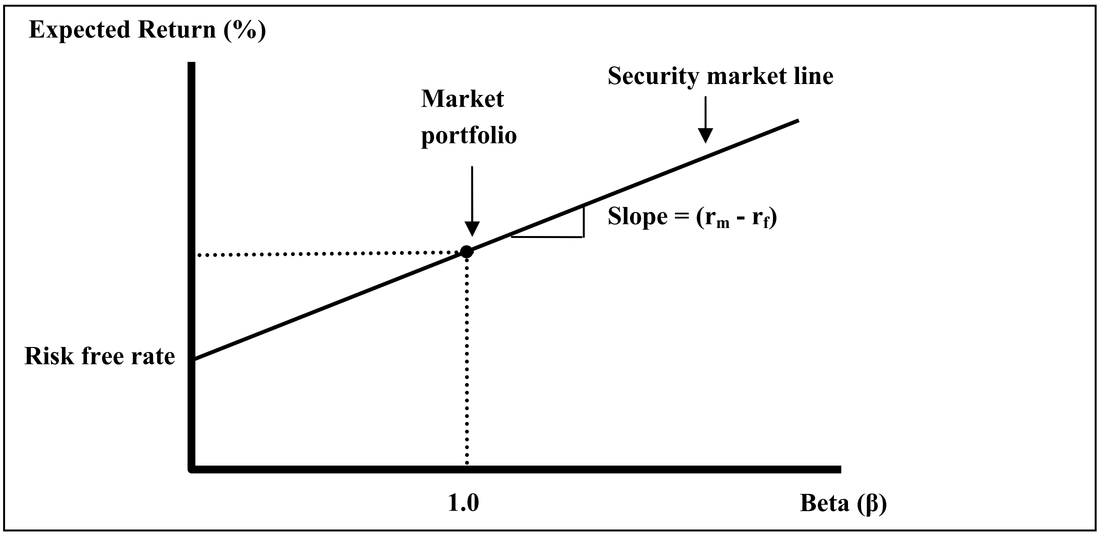
The **C**apital **A**ssets **P**ricing **M**odel (CAPM) derives the expected return on an assets in a market, given the risk-free rate available to investors and the compensation for market risk. CAPM specifies that the expected return on an asset is a linear function of its beta and the market risk premium:

(32) $$\text{Expected return on stock i} = r_i = r_f + \beta_i (r_m - r_f)$$

Where r_f is the risk-free rate, β_i is stock *i*'s sensitivity to movements in the overall stock market, whereas ($r_m - r_f$) is the market risk premium per unit of risk. Thus, the expected return is equal to the risk free-rate plus compensation for the exposure to market risk. As β_i is measuring stock *i*'s exposure to market risk in units of risk, and the market risk premium is the compensations to investors per unit of risk, the compensation for market risk of stock *i* is equal to the β_i ($r_m - r_f$).

Figure 6 illustrates CAPM:

Figure 6: Portfolio expected return



The relationship between β and required return is plotted on the *securities market line*, which shows expected return as a function of β. Thus, the security market line essentially graphs the results from the CAPM theory. The x-axis represents the risk (beta), and the y-axis represents the expected return. The intercept is the risk-free rate available for the market, while the slope is the market risk premium ($r_m - r_f$)

CAPM is a simple but powerful model. Moreover it takes into account the basic principles of portfolio selection:

1. Efficient portfolios (Maximize expected return subject to risk)
2. Highest ratio of risk premium to standard deviation is a combination of the market portfolio and the risk-free asset
3. Individual stocks should be selected based on their contribution to portfolio risk
4. Beta measures the marginal contribution of a stock to the risk of the market portfolio

CAPM theory predicts that all assets should be priced such that they fit along the security market line one way or the other. If a stock is priced such that it offers a higher return than what is predicted by CAPM, investors will rush to buy the stock. The increased demand will be reflected in a higher stock price and subsequently in lower return. This will occur until the stock fits on the security market line. Similarly, if a stock is priced such that it offers a lower return than the return implied by CAPM, investor would hesitate to buy the stock. This will provide a negative impact on the stock price and increase the return until it equals the expected value from CAPM.

5.7 Alternative asset pricing models

5.7.1 Arbitrage pricing theory

Arbitrage pricing theory (APT) assumes that the return on a stock depends partly on macroeconomic factors and partly on noise, which are company specific events. Thus, under APT the expected stock return depends on an unspecified number of macroeconomic factors plus noise:

(33) $$\text{Expected return} = a + b_1 \cdot r_{factor\ 1} + b_2 \cdot r_{factor\ 2} + \ldots + b_n \cdot r_{factor\ n} + noise$$

Where $b_1, b_2, \ldots, b_n$ is the sensitivity to each of the factors. As such the theory does not specify what the factors are except for the notion of pervasive macroeconomic conditions. Examples of factors that might be included are return on the market portfolio, an interest rate factor, GDP, exchange rates, oil prices, etc.

Similarly, the expected risk premium on each stock depends on the sensitivity to each factor ($b_1, b_2, \ldots, b_n$) and the expected risk premium associated with the factors:

(34) $$\text{Expected risk premium} = b_1 \cdot (r_{factor\,1} - r_f) + b_2 \cdot (r_{factor\,2} - r_f) + \ldots + b_n \cdot (r_{factor\,n} - r_f)$$

In the special case where the expected risk premium is proportional only to the portfolio's market beta, APT and CAPM are essentially identical.

APT theory has two central statements:

1. A diversified portfolio designed to eliminate the macroeconomic risk (i.e. have zero sensitivity to each factor) is essentially risk-free and will therefore be priced such that it offers the risk-free rate as interest.
2. A diversified portfolio designed to be exposed to e.g. factor 1, will offer a risk premium that varies in proportion to the portfolio's sensitivity to factor 1.

5.7.2 Consumption beta

If investors are concerned about an investment's impact on future consumption rather than wealth, a security's risk is related to its sensitivity to changes in the investor's consumption rather than wealth. In this case the expected return is a function of the stock's consumption beta rather than its market beta. Thus, under the consumption CAPM the most important risks to investors are those the might cutback future consumption.

5.7.3 Three-Factor Model

The three factor model is a variation of the arbitrage pricing theory that explicitly states that the risk premium on securities depends on three common risk factors: a market factor, a size factor, and a book-to-market factor:

(35) $$\text{Expected risk premium} = b_{market} \cdot (r_{market\ fa\cot r}) + b_{size} \cdot (r_{size\ factor}) + b_{book-to-market} \cdot (r_{book-to-market})$$

Where the three factors are measured in the following way:

- Market factor is the return on market portfolio minus the risk-free rate
- Size factor is the return on small-firm stocks minus the return on large-firm stocks (small minus big)
- Book-to-market factor is measured by the return on high book-to-market value stocks minus the return on low book-value stocks (high minus low)

As the three factor model was suggested by Fama and French, the model is commonly known as the Fama-French three-factor model.

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